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THE DEVELOPMENT OF LAYERED THERMOSOLUTAL CONVECTION

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Abstract—When a solution containing a non-linear profile of solute concentration is heated from below, convection can occur in layers. Observations of this phenomenon are presented, and these are compared with calculations based on a simple model. It is concluded that the model contains the essential features of the process.

NOMENCLATURE

D,	solute diffusion coefficient;
D_t,	convective solute transfer coefficient;
g,	acceleration due to gravity;
h,	depth of the convecting layer;
K,	thermal diffusivity;
K_t,	convective heat transfer coefficient;
P,	the Prandtl number of the liquid, $= \nu/K$;
R,	the thermal Rayleigh number (positive destabilizing), $= \dfrac{g\Delta\rho_t h^3}{\nu K\rho}$;
R_s,	the solutal Rayleigh number (positive destabilizing), $= \dfrac{g\Delta\rho_s h^3}{\nu K\rho}$;
R_e,	a thermosolutal Rayleigh number, $= R + \dfrac{P}{P+1}R_s$;
R_{ec},	the critical value of R_e;
ν,	kinematic viscosity;
ρ,	a characteristic density of the liquid;
$\Delta\rho_t$,	the thermal component of the density difference between top and bottom of the convecting layer;
$\Delta\rho_s$,	the solutal component of the density difference between top and bottom of the convecting layer.

1. INTRODUCTION

PURE-THERMAL convection occurs in a homogeneous fluid when a sufficient spatial temperature variation is imposed on it. A considerable amount of experimental and theoretical work has been devoted to this phenomenon, both to define the conditions for marginal stability, when convection is about to begin, and also to achieve an understanding of the regime in which velocities are large. In certain situations, particularly in oceanography and limnology, the fluid is not homogeneous, but contains a solute whose concentration varies vertically. Since the density of the fluid is affected by the solute as well as by the heat, and since these two diffuse at different rates, the resulting thermosolutal convection differs in certain important respects from pure-thermal convection. These differences are exhibited both at marginal stability and at finite amplitude; the question of marginal stability has been considered elsewhere [1, 2], and this paper is concerned with finite-amplitude effects.

The first experimental study of thermosolutal convection was that of Turner and Stommel [3]. In this experiment a container of water, which contained dissolved salt with a stabilising gradient of concentration, was heated gently from below. By means of suspended aluminium powder and fluorescein dye it was seen that the resulting convection occurred in layers. The layers were formed in turn starting from the bottom, and they slowly increased in thickness after formation. This observation inspired the theoretical study of Veronis [4], which revealed a number of the properties of steady-state thermosolutal convection, but which failed to explain the observed layering. Schaafs [5] observed the phenomenon in various liquids containing stable solute gradients when they were irradiated by light from above, but again no explanation was given.

Turner [6] carried out an experiment which was designed to elucidate the characteristics of stratiform thermosolutal convection. Instead of allowing the layers to appear in an uncontrolled manner, he deliberately produced a two-layer system. Having set up a sharp boundary between the strong and the weak solutions by stirring each, he studied the rate at which salt and heat were transferred across the boundary as a function of the ratio of the contributions of salt and heat to the density difference. The characteristics of even this apparently simple system were not immediately understandable, and the results were published in graphical form without explanation. No full description of the system has yet been published; however a tentative theory [7] has shown that the results are consistent, to a reasonable approximation, with the following properties: (i) The convecting layers are separated by a thin region in which heat and solute are transported by diffusion only, no convection penetrating the region; (ii) the non-dimensional rate at which solute is lifted through a convecting layer is proportional to the non-dimensional upward heat flow, the solutal Nusselt number being perhaps $O(10^2)$ times the thermal Nusselt number; (iii) the upward heat flow is described by an effective Rayleigh number R_e in the same way as the thermal Rayleigh number R describes the convective heat flow in pure-thermal convection (here $R_e = R + (P/P + 1) R_s$, where P is the Prandtl number of the liquid and R_s is the solutal Rayleigh number; $|R/R_s|$ is the ratio of the contributions of heat and solute to the density difference across the convecting layer); and (iv) the profiles of temperature and solute concentration adjust themselves according to a maximum criterion such as maximum heat flow or maximum R_e.

2. THE FORMATION OF LAYERS

These observations concerning the probable nature of the steady state stratiform convection in Turner's system do not reveal why the Turner–Stommel layers should form in the first place; steady-state pure-thermal convection generally occurs as a flow which occupies the whole depth of the fluid layer. In fact, the first layer seems to form because of non-linearity of the concentration profile [2]. A typical experiment would be carried out in a container whose base was made of some substance which was impervious to solute but not to heat (e.g. brass). As a result the gradient of concentration would be zero at the bottom of the liquid, so that heating from below would produce a density gradient which was potentially unstable there. Further up, the stabilising effect of the concentration profile would exceed the destabilising effect of the temperature profile; instability would therefore be confined to a thin layer at the bottom of the container. Observations, and the theory of Veronis [4], suggest that the first effect of the convection in this layer is to carry solute upwards relatively quickly to form a sharp, stable, gradient of density at the top of the layer. As the concentration becomes more uniform throughout the layer, the heat transfer increases, until finally concentration and temperature are both uniform except near the boundaries.

In considering the initiation of the second layer, it is important to remember that the diffusion constant for solute is much less than that for heat, at least for the aqueous solutions so far used for these experiments. Above the convecting layer, heat and solute are initially transported only by diffusion. The supply of heat and solute to this diffusing region would be greatly increased by the development of convection in the first layer. But whereas the consequent build-up of heat at the top of the layer would diffuse quite rapidly into the stable region, the solute would be confined to the boundary for a much longer period. As a result, there would be a region above the first convecting layer which would experience a transient increase of destabilising temperature gradient without a corresponding increase of stabilizing concentration gradient, which could lead to instability in this region and the formation of a second convecting layer. Other layers could follow by the same process.

3. OBSERVATIONS

Figure 1 shows a sequence of records obtained during one of a number of experiments similar to that of Turner and Stommel. A solution of sugar in water, the concentration of which decreased upwards, was held in a glass-sided tank whose internal dimensions were: length 25 cm, width 6·4 cm, depth (of liquid) 9·7 cm. Conditions in the liquid were monitored optically by what was basically a conventional schlieren system. The addition of a scanning mechanism also allowed quantitative observations, however, and the system could be converted rapidly from one mode to the other. The main penalty incurred for this flexibility was that the usual knife-edge was replaced by a vertical series of slits, which led to the background illumination of the image being confined to a series of horizontal strips. Since the middle slit was rather wide, this did not in practice obscure the required detail.

Schlieren observation depends on a variation of refractive index of the liquid. In the case of aqueous sugar solutions, the refractive index is a function of both temperature and concentration, but in such a way as to be determined almost entirely by the resulting density. Hence in the presence of variations of both temperature and concentration, the schlieren observations give information on the distribution of the resulting gradients of density, irrespective of which influence is predominant. The initial distribution of solute in the solution was determined by a measurement shortly before heating began, when the fluid was isothermal and the refractive index varied with concentration only. Thus each curve in Fig. 1(a) is a graph of concentration gradient versus depth; since there is no diffusion through the top and bottom boundaries, the zero of each curve is the line joining its end points. The scale is given by the superimposed grid, adjacent points of which are separated by 0·0028 g cm^{-4} in density gradient, or 2 cm in depth.

The base of the tank was a brass plate 1·25 cm thick, which rested on a bath of oil which was electrically heated. Figures 1(b–n) are a sequence of frames exposed after heating of the solution represented in Fig. 1(a) began. Figures 1(b) and 1(c) show the cellular structure of the convection in the first layer at an early stage, and the way in which the depth of this layer increased with time. In Fig. 1(d) the second layer may just be seen; Fig. 1(e) is the corresponding density gradient profile. Once convection began the density gradient trace became somewhat obscured, but in Fig. 1(e) and, particularly, later profiles, the main features are clear: each layer tended to a state of zero density gradient after convection developed in it; adjacent layers were separated by a sharp peak of stable density gradient; a similar peak also occurred above the uppermost layer, and above this peak the density gradient was progressively reduced toward zero. These features are qualitatively in accord with the mechanism proposed for the formation of the second and later layers.

Figure 1(n) was exposed 95 min after the heating current was switched off. Since the temperature profile decays by diffusion much more rapidly than the concentration profile, this frame is substantially indicative of the distribution of solute at the end of the experiment.

The foregoing description of the layering process takes no account of horizontal variations of the heat and solute flows. Examination of the sequence in Fig. 1 shows that once a convecting layer has developed past the initial stage of having a clear cellular structure, its boundaries remain planar and horizontal to a reasonable approximation. Nevertheless departure from this condition of horizontal constancy does occur, particularly in the early stage when a layer is just forming or about to form. The profiles in Fig. 1 show this effect, which possibly modifies the formation of new layers. In each profile, there are five curves recorded simultaneously, displaced laterally about 0·4 cm apart across the tank image. In each case, in the region where a new layer is forming, these show a variation of conditions across the field of view, the density gradient showing more marked maxima and minima on one side than the other.

This effect may result from a variation of vertical heat flow across the tank caused, for example, by cellular flow in the bottom layer. Thus the layers and boundaries may form in accordance with the postulated mechanism where the heat flow is high, but only approximately obey it elsewhere. Alternatively, the effect may result from the presence of internal waves, excited by the underlying motion. In this latter case the formation of a new layer may be somewhat modified by the motion and the possible associated mixing.

Standing internal waves would represent a limitation of the laboratory experiments, since they are maintained by the walls of the tank. Their effect is likely to diminish as the size of the vessel increases, and in a lake or ocean they might have no significant influence. It is likely that they were not too important in these experiments, since the layer development was in accordance with expectation, at least qualitatively; but they provide an element of doubt as to the extent to which quantitative analysis is justified. As a result, no such direct analysis has been attempted. Instead, a numerical model of the postulated mechanism was developed. This was found to give quite similar results to those obtained in the experiments, which makes it likely that this mechanism is basically correct.

4. THE NUMERICAL MODEL

The formation of the first convecting layer can be described in terms of a known heat flux supplied to the tank, and a known preexisting distribution of solute [2], but the onset of convection in subsequent layers is more complex. Presumably a similar criterion for marginal stability in terms of thermal and solutal Rayleigh numbers applies as to the first layer, but the fluxes of heat and solute are now uncertain, being due in part to the convection in the adjacent layer. To predict accurately the development of a series of layers would therefore be very difficult, both because the layers are interdependent, and also because no satisfactory theory is available to describe the dependence of the solute and heat fluxes on R and R_s. It is therefore necessary to seek a limited representation of the system which is sufficiently tractable to allow calculation.

The model to be described here is a crude one, which was not expected to yield accurate results. Nevertheless it reproduced the behaviour of the laboratory experiments well enough for the similarity to be apparent, and it certainly showed qualitatively the formation of layers. Its basis was the representation of both diffusive and convective transfer of solute and heat as diffusion, the relevant diffusion constant being increased in an appropriate way when convection was present. The problem was thus converted to one of diffusion in one dimension (vertically) through a series of slabs of variable diffusivity and with mobile (horizontal) boundaries. In this form it was amenable to numerical, though not analytical, solution. Any properties of the real system which depend on horizontal irregularities were lost in the model; the degree to which the model is successful therefore gives an indication of the importance of these effects.

The model operated as follows. The basic information required was a concentration gradient profile, in the form of values at a number of equally-spaced levels in the tank. This was subjected to harmonic analysis and integrated, so that at any level in the liquid a value could be found for the difference in concentration between that level and the bottom of the liquid.

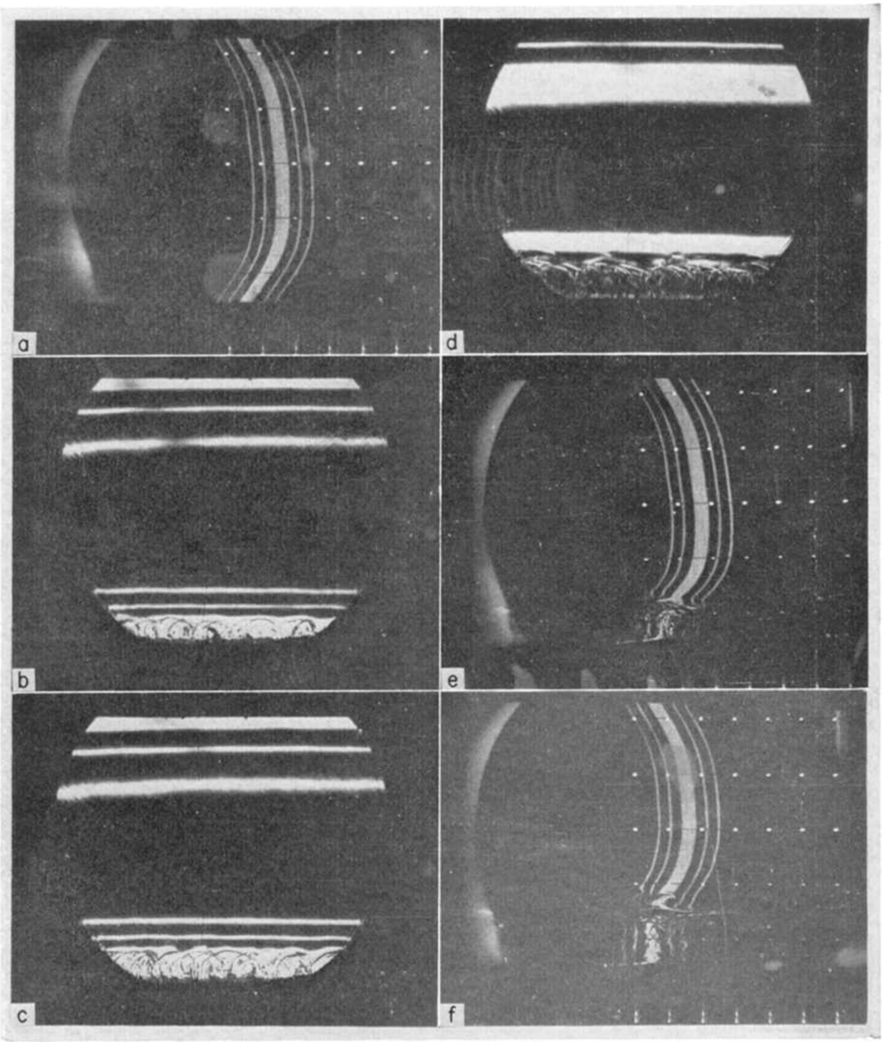

FIG. 1. (a–f).

FIG. 1. A sequence of schlieren and scanning-schlieren observations during the development of layered thermosolutal convection in a solution of sugar in water. (a) The profile of density gradient before heating began, i.e. that due to the profile of solute concentration only. In this and the other qualitative observations adjacent points of the superimposed grid are separated 2 cm vertically and $0\cdot0028$ g cm^{-4} horizontally. (b) 23 min after heating began. In this and the other qualitative observations the background illumination is a series of horizontal strips. These correspond to the series of traces present in each of the quantitative observations. (c) 27 min. (d) 32 min. (e) 34 min. (f) 53 min. Note the development of a stable density gradient above the well-developed convecting layer. (g) 62 min. (h) 63 min. (i) 79 min. (j) 90 min. (k) 91 min. (l) 112 min. (m) 139 min. (n) 222 min after heating began, and 95 min. after the heating was switched off. This frame is substantially indicative of the distribution of solute at the end of the experiment.

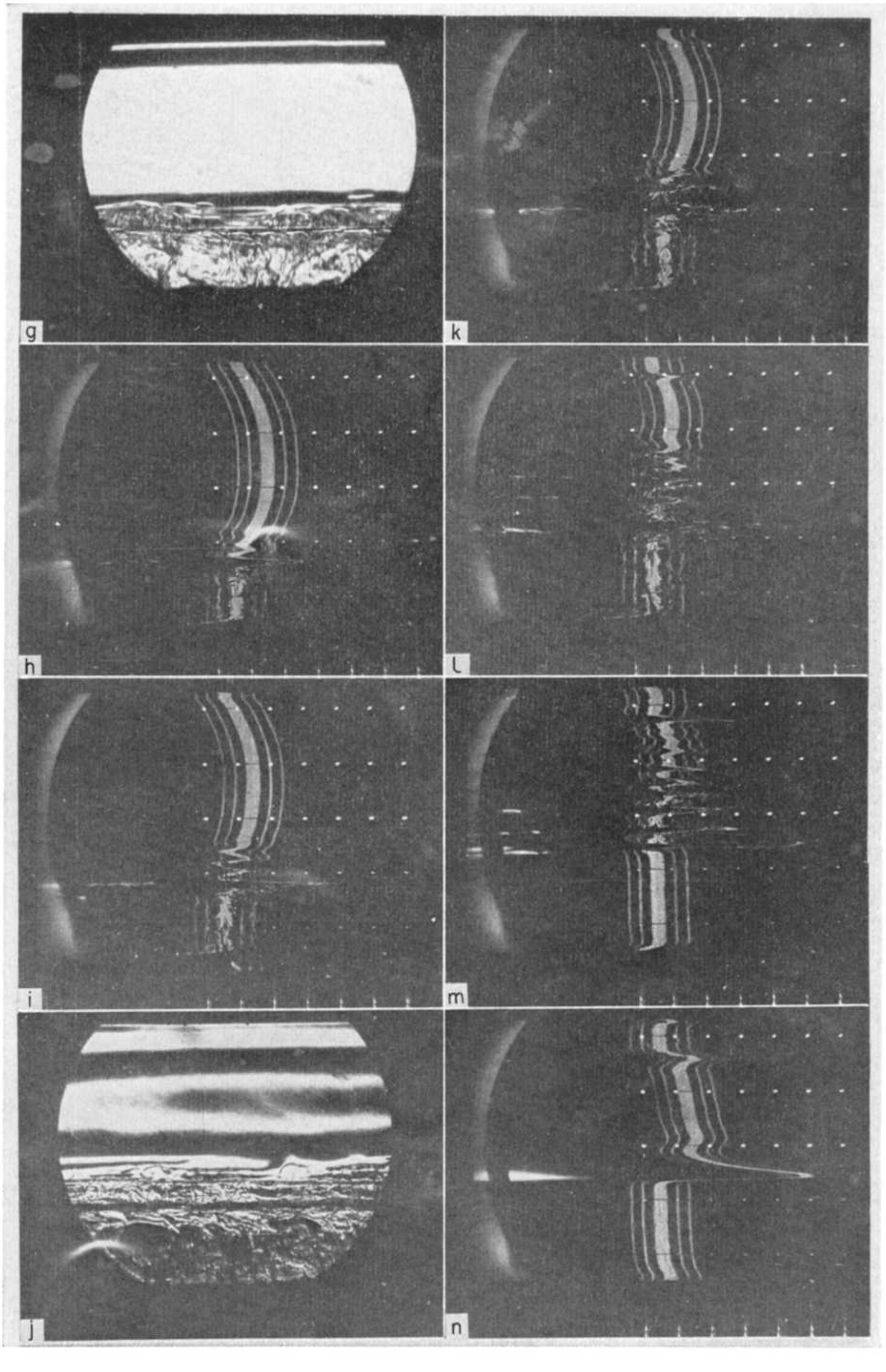

FIG. 1 (g–n).

In addition, an initial temperature, and the coefficients of an expression defining the growth of temperature with time at the bottom boundary, were required. The initial temperatures and concentrations at a large number (usually 200) of equally spaced levels in the "tank" were thus calculated. The development of these values with time was then calculated for successive equal time increments by a simple numerical scheme for a small number (usually 10) of time steps. This simple scheme was stable only for small time increments, but gave a new profile in terms of the immediately preceding one only, and was a useful "starting" scheme. The "running" scheme was based on a numerical solution of the heat-conduction equation due to Du Fort and Frankel [8], and involved both the two preceding profiles. This scheme was stable for large time increments (or large diffusivities) and was therefore more suitable for the calculation of conditions after the onset of convection.

In accordance with the interpretation of Turner's experiment (conclusion (iii)) mentioned in the Introduction, the heat-transfer coefficient for a convecting layer was assumed to be $K_t(R_e)$, where $K_t(R)$ is the heat-transfer coefficient in pure-thermal convection. Heat flow measurements in the pure-thermal case have been made by Malkus [9], who found that the convective heat transport varies linearly with Rayleigh number; but that there are several discrete transitions in the character of the convection, at each of which the slope of the line changes, increasing with increasing heat flow. These experiments have been variously interpreted by Malkus [10], Malkus and Veronis [11], and Elder [12]. However we are concerned here with the actual results rather than their explanation, and these have been substantially confirmed by later measurements by Thomas and Townsend [13] and Globe and Dropkin [14]. It is not difficult to find an empirical relationship which represents these measurements tolerably well; the main requirements are that the heat-transport coefficient must (a) be continuous, (b) be equal to the molecular value when the heat flow is less than or equal to that required to induce marginal convective stability, and (c) vary as the one-third power of the Rayleigh number when the latter is high. The expression which was thus derived for $K_t(R_e)$ was

$$\left.\begin{aligned} R_e > R_{ec}:&\quad K_t = K\,[1 + 0{\cdot}72\,(R_e/R_{ec})^{\frac{1}{3}} \\ &\qquad\qquad\times (1 - R_{ec}/R_e)] \\ R_e \leqslant R_{ec}:&\quad K_t = K \end{aligned}\right\} \quad (1)$$

Here K is the molecular value of the thermal diffusivity, and R_{ec} is the critical value of R_e, for which stability is marginal. The pure-thermal form of (1), in which R and R_c replace R_e and R_{ec}, matches the results of Malkus within 3 per cent at all the transition points, and both forms satisfy the above requirements (a)–(c). It must be noted here that (1) describes the heat transfer in the steady state only. It will be seen later that it had to be modified in order to take account of time-dependent effects.

Once the running solution had been established, the profiles were scanned after each pair of time increments to determine whether R_e was positive (i.e. potentially unstable) between any two neighbouring levels. Starting at any such point, the upper and lower bounds of a search layer were extended until R_e was maximized. Then if this maximum value exceeded R_{ec}, all the levels within the layer were assigned a new K_t and D_t (where D_t is the solute transfer coefficient).

The question of how to assign the value of D_t was decided by experiment, bearing in mind that the effect of convection on the solute transfer rate is initially much larger than the corresponding thermal effect. In modelling tank experiments it was found satisfactory to write:

$$\left.\begin{aligned} R_e > R_{ec}:&\quad D_t = K_t - K + D \\ R_e \leqslant R_{ec}:&\quad D_t = D \end{aligned}\right\} \quad (2)$$

Since $K \sim 10^2 D$, the fractional increase in transfer rate of solute was about 100 times that of heat when convection began. This lies between the limits of 1 and 10^4 predicted by the theory of Veronis [4] for large- and small-amplitude convection respectively.

5. RESULTS

The results of a run in which the conditions modelled those of the experiment from which Fig. 1 was obtained are shown in Figs. 2, 3 and 4. The graphs of concentration and temperature in Figs. 2 and 3 show clearly the way in which an initially smooth density profile is broken into steps by the convection, the density contrast between adjacent steps being maintained by the low diffusion constant of the solute in the non-convecting regions. The concentration profiles also show the way in which two layers may amalgamate. The two layers present up to 79 min have a steadily decreasing difference of concentration. By 91 min this has diminished to the point at which convection has occurred over the whole region, speedily extinguishing any remaining contrast.

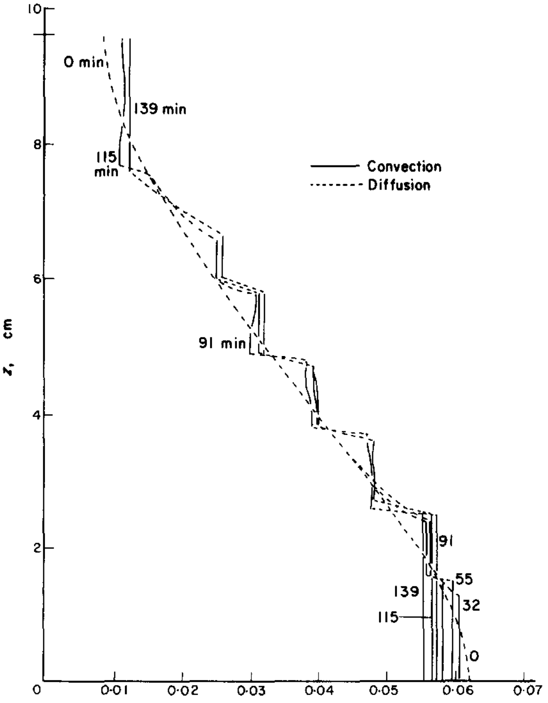

FIG. 2. Concentration profiles calculated according to the model. The initial concentration gradient profile used was that shown in Fig. 1(a), and the heating rate was that which obtained during the experiment from which the other observations of Fig. 1 were drawn. The profiles were calculated for the times indicated after heating began.

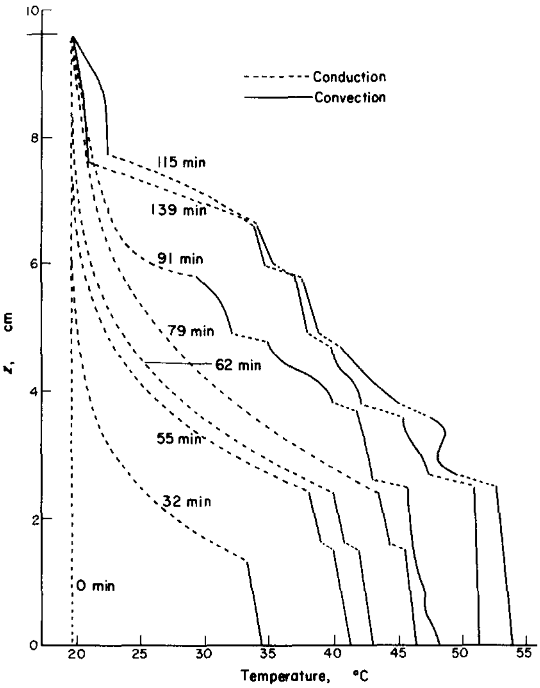

FIG. 3. Temperature profiles calculated according to the model, as for Fig. 2.

It is clear from the graphs that, while the numerical scheme is apparently stable, it is not free of errors, which seem to arise mainly when diffusion constants are reassigned. Such errors are shown particularly by the density gradient profiles, where their magnitudes are emphasized by the differentiation. They do not seem serious enough to affect the results significantly.

A further limitation of the method, not shown by these graphs, is that the time-development of the concentration and temperature profiles within a convecting layer is not well represented. Diffusion in a system of scale length h and diffusion constant X has a characteristic time

$t = h^2/X$. The time development of pure-thermal convection has been studied for a particular configuration by Foster [15], and the results show characteristic times of order $10^{-3}\ h^2/X$, where X is here the thermal diffusivity. It is not possible to deduce a general expression from Foster's results; but it seems very likely that not only is the characteristic time for the growth of convective disturbances very short, but that also it depends on h in a manner different from that appropriate to diffusion. Now K_t in the model was chosen initially to provide the correct heat transfer rate through the convecting layers in the steady state. Foster's results show that this choice is incompatible with a realistic representation of the time-development of the profile within a layer, and furthermore it seems likely that the degree of incompatibility depends on the scale of the system.

To meet this difficulty, the dependence of K_t on R_e in the model was increased until reasonable results were obtained. The graphs shown in Figs. 2–4 were obtained by increasing by a factor 10 the convective component of K_t and D_t, that is,

$$R_e > R_{ec}: \quad K_t = K[1 + 7{\cdot}2\,(R_e/R_{ec})^{\frac{1}{3}} \times (1 - R_{ec}/R_e)]$$

$$R_e \leqslant R_{ec}: \quad K_t = K$$

with D_t evaluated according to (2).

While the agreement between the experimental and the model results is quite close, the difficulties discussed above limit the quantitative validity of the method. Although a continued search for better ways of evaluating K_t and D_t, and longer computations using a finer mesh spacing, would yield more impressive results than those exhibited, the limit of credibility has probably been reached. The important

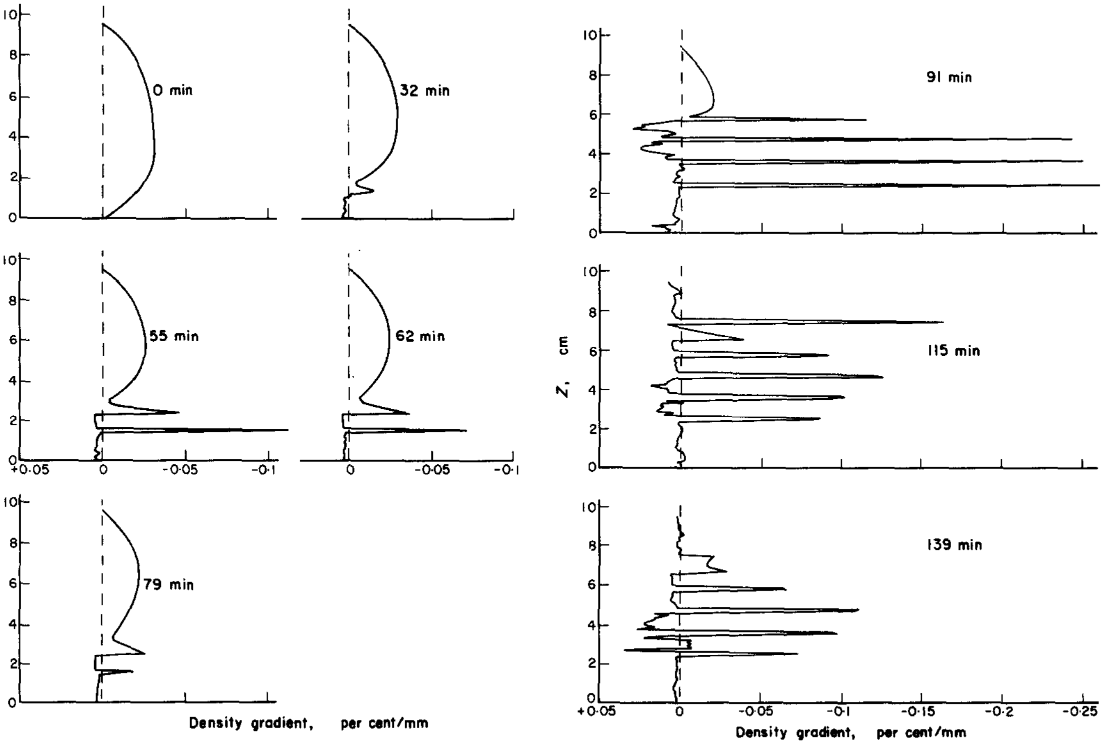

FIG. 4. Density gradient profiles calculated according to the model, as for Fig. 2.

points are (i) that the problems of quantitative accuracy of the model are not sufficiently important to prevent a meaningful comparison of the model results with those of the corresponding experiment, (ii) that the model and experimental results agree sufficiently closely to show that the model is an adequate description of the process, and that the effects of horizontal irregularities are not important, and (iii) that the model is based on a simple description of the convective process entirely in terms of Rayleigh numbers, but reproduces the behaviour observed in experiments quite well. To the extent that this description is valid, the dimensionless character of the Rayleigh numbers removes any problems of scale, and the model method may be applied to a system of any size.

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Résumé—Lorsqu'une solution contenant un profil non-linéaire de concentration du soluté est chauffé par en-dessous, la convection peut se produire sous forme de couches. On présente des observation de ce phénomène, et celles-ci sont comparées avec des calculs basés sur un modèle sur un modèle simple. On en conclut que le modèle contient les caractéristiques essentielles du processus.

Zusammenfassung—Wird eine Lösung mit nicht-linearem Konzentrationsprofil von unten beheizt, so so kann eine Konvektion in Schichten erfolgen. Beobachtungen dieses Phänomens werden gezeigt und mit Berechnungen an einem einfachen Modell verglichen. Es wird angenommen, dass das Modell alle wesentlichen Vorgänge dieses Problems umfasst.

Аннотация—При нагреве снизу раствора с нелинейным профилем концентрации растворенного вещества может возникать конвекция в слоях. Приводятся результаты наблюдений этого явления, которые сравниваются с расчетами, проделанными на простой модели. Делается вывод, что выбранная модель согласуется с существенными особенностями этого процесса.